Image Segmentation in the Framework of Projective Morphology[1]

Y. Vizilter, S. Zheltov[2]

[2] State Research Institute of Aviation Systems, Russia, 125319, Moscow, Viktorenko street, 7. Ph.:+7(499) 157-94-98, e-mail: viz@gosniias.ru, zhl@gosniias.ru

Morphological image segmentation is addressed within the framework of projective morphology. Schemes for design of morphological projectors based on segmentation procedures with and without the loss of information are proposed. Projective segmentation without loss is based on a set of equivalent transform rules. Segmentation with loss is projective at least for two main classes of morphological operators: minimum-distance and monotonous.

Introduction

Image segmentation in the wide sense means extraction of any structural image information relative to some given image models. Segmentation in the narrow sense presumes the tessellation of image into a set of connected regions of homogeneous intensity or texture. Image segmentation task is very close to image compression – both in a problem statement and in solution structure. Two main criterions control segmentation and image compression techniques: minimal size of image description and maximal precision of reconstruction based on this description.

In [1] the so-called *projective morphology* was introduced. In this paper concepts and schemes of projective morphology are applied to morphological image segmentation.

Projective morphologies and the segmentation problem statement

The *projective space of patterns* (images) is as an algebraic system $<\Psi, \Omega, \bullet, \mathrm{V}, \mu, Pr, \boldsymbol{E}>$, where Ψ is a set of *scalars* including 0 and 1; Ω is the set of *patterns* with "zero pattern" $\varnothing$; '$\bullet$' is the multiplicative group operation of *multiplication of scalars* $\Psi \times \Psi \to \Psi$ and a *scalar by pattern multiplication* $\Psi \times \Omega \to \Omega$; '$\mathrm{V}$' $\in$ {'+', '×', '∪', '∩', '∨', '∧', '*min*', '*max*', …} is the additive Abel semi-group of *scalars fusion* $\Psi \times \Psi \to \Psi$ and *patterns fusion* $\Omega\times\Omega\to\Omega$; μ is the *norm* of the pattern $\Omega \to \mathrm{R}$ ($\mu(A) = \|A\|$, $\|\varnothing\| = 0$); set of basic patterns (*primitives*) $\mathbf{E} = \{E_1, \ldots, E_n\}$ is the basis of the *morphological pattern decomposition.* Let $\boldsymbol{E}$ will denote the corresponding *morphological subspace* $\boldsymbol{E} \subseteq \Omega$ generated by the algebraic closing of basis $\mathbf{E}$ relative to'$\bullet$,V'-combination.

The operator of *linear projection of pattern onto the pattern* has a form

$$Pr(A,B) = r(A,B)\bullet B\text{: } \Omega \to \boldsymbol{B} \subseteq \Omega,$$

where $r(A,B) \in \Psi$ is the *coefficient of linear connection* of pattern A relative to pattern B. The projection of *pattern onto subspace* has a form

$$Pr(A,\boldsymbol{E}) : \Omega \to \boldsymbol{E} \subseteq \Omega.$$

This projection operator satisfy, first, the *projective property*

$$Pr(A,\boldsymbol{E}) = Pr(Pr(A,\boldsymbol{E}),\boldsymbol{E}),$$

and, second, *decomposition condition*

$$Pr(A,\boldsymbol{E}) = \mathrm{V}_{k = 1,\ldots,n}(Pr(A,E_k)) = \mathrm{V}_{k = 1,\ldots,n}(r(A,E_k)\bullet E_k),$$

where $\mathbf{a}(A,\boldsymbol{E}) = <r(A,E_k)>$ is the *vector of morphological decomposition* of pattern A in basis $\boldsymbol{E}$. *Morphological decomposition* is a mapping

$$\mathbf{dec}_{\mathbf{E}}(A) = <r(A,E_1), .., r(A,E_n)>: \Omega\to\Psi^n \qquad (1)$$

Let us state the problem of the morphological image segmentation based on notions listed above. Let some basis $\mathbf{X}$ of decomposition is *complete* on Ω: $\forall A \in \Omega$: $Pr(A,\boldsymbol{X}) = A$.

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This decomposition is *complete* for reconstruction, so, the *linear closure* of the basis $\mathbf{X}$ is complete: $\boldsymbol{X} = \Omega$.

Let *morphological descriptor* of any pattern A will be some fixed structure of the form

$$\mathbf{d}(A,\boldsymbol{E}) = \{n, d(A,E_1), \ldots, d(A,E_n)\}, \qquad (2)$$

where $\mathbf{E} = \{E_1, \ldots, E_n\} \in \Omega^n$ is a basis of the morphological decomposition; $n = \dim(\mathbf{E})$ is the dimension of the basis $\mathbf{E}$. $d(A,E_i)$ is the *descriptor of an element* of the decomposition. *Descriptor size* $\nu(\mathbf{d})$ is just a memory size for storing the descriptor $\mathbf{d}$ (2). If all primitives have the same type, then *descriptor size* $\nu(\mathbf{d})$ is proportional to basis dimension, and we can consider it as a *descriptor dimension*.

Let some complete basis $\mathbf{X}$ is given. Then for any $A \in \Omega$ its descriptor $\mathbf{d}(A,\mathbf{X})$ is a *complete descriptor*. Any *subbasis* $\mathbf{Y} = \{Y_1, \ldots, Y_m\}$: $\boldsymbol{Y} \subseteq \boldsymbol{X}$, $\dim(\mathbf{Y}) \leq \dim(\mathbf{X})$ determines a *subdescriptor* $\mathbf{d}(A,\mathbf{Y})$. The set $\Theta(\mathbf{X}) = \{\mathbf{d}(A,\mathbf{Y}):\ A \in \Omega,\ \boldsymbol{Y} \subseteq \boldsymbol{X}\}$ is a *set of all subdescriptors* based on $\mathbf{X}$.

In the framework of *shape theory* [2] the *morphological segmentation operator* can be introduced a mapping of pattern from Ω to some subdescriptor from $\Theta(\mathbf{X})$

$$\varepsilon_S: \Omega \to \Theta(\mathbf{X}) \qquad (3)$$

The operator of morphological reconstruction will be an operator

$$\delta_S: \Theta(\mathbf{X}) \to \Omega.$$

So, the *morphological filter*

$$Pr(A,\Theta(\mathbf{X})) = \psi_s A = \delta_s \varepsilon_s A: \Omega \to \Omega \qquad (4)$$

will be an algebraic projector

$$\psi_s^2 = \psi_s. \qquad (5)$$

Let us define a *segmentation cost function* in the form of a functional criterion $\Phi(A,\mathbf{Y})$ that establishes a penalty both for the descriptor size ν and for the deviation of the segmented pattern from the initial pattern (*reconstruction quality criterion J*):

$$\Phi(A,\mathbf{Y}) = J(A,Pr(A,\boldsymbol{Y})) + \alpha \times \nu(\mathbf{d}(A,\mathbf{Y})),$$

$$\Phi(A,\mathbf{Y}) \to min(\mathbf{Y}: \boldsymbol{Y} \subseteq \boldsymbol{X}), \qquad (6)$$

where α is a tuning parameter. So, the *problem of optimal morphological segmentation* consists in finding an *optimal subbasis* $\mathbf{Y}$ that provides a minimum to functional criterion (6). The *optimal segmentation* S can be defined as

$$S(A,\mathbf{X}) = argmin_{\boldsymbol{Y}} \{\Phi(A,\mathbf{Y}): \boldsymbol{Y} \subseteq \boldsymbol{X}\}. \qquad (7)$$

Such transition from complete model $\boldsymbol{X}$ to *sub-model* $\boldsymbol{Y} \subseteq \boldsymbol{X}$ means "*shape simplification*" in a Pytiev morphological sense [3]. Moreover, since $\boldsymbol{Y} \subseteq \boldsymbol{X}$, then

$Y_1 = \mathrm{V}_{k=1,\ldots,n}(s_{1k} \bullet X_k), \ldots, Y_m =$
$= \mathrm{V}_{k=1,\ldots,n}(s_{mk} \bullet X_k)$.

This means that the segmentation procedure S can be described as a *matrix of transition to new basis* $[\mathbf{S}(A)]$ of dimension $m \times n$

$$\mathbf{Y} = [\mathbf{S}(A)]\mathbf{X},\ [\mathbf{S}(A)] = \| s_{lk} \|$$

$$l = 1, \ldots, m;\ k = 1, \ldots, n. \qquad (8)$$

Thus the segmentation problem is reduced to finding the matrix of '$\bullet$,V'-linear transformation (8) according to criterion (6).

Projective morphologies based on segmentation without loss of information

Any two bases $\mathbf{X}$ and $\mathbf{Y}$ such that

$$\mathbf{Y} \sim_A \mathbf{X} \Leftrightarrow Pr(A,\boldsymbol{Y}) = Pr(A,\boldsymbol{X})$$

can be called *equivalent relative to pattern* A. For such subbasis procedure (7) takes the form

$$S(A,\mathbf{X}) = argmin_{\boldsymbol{Y}} \{\nu(\mathbf{d}(A,\mathbf{Y})): \boldsymbol{Y} \subseteq \boldsymbol{X},\ Pr(A,\boldsymbol{Y}) = A\},$$

therefore $Pr(A,\boldsymbol{Y})$ is a *non-distorting projector with minimum-size descriptor*. Let us define the following *dimension reducing equivalent transformations* of bases:

1) *elimination* of zero-coefficient primitives;
2) *grouping* of equal-coefficient primitives.

Elements of corresponding transition matrix $[\mathbf{S}(A)]$ take values in a set $\{0,1\}$. In practice it is not reasonable to group *all* elements with the same weights. For each particular morphological system there are some special constraints on admissible grouping of primitives. These constraints can be described by *transition rules predicate*

$\mathrm{p}(\mathbf{X} \to \mathbf{Y}) = \mathrm{p}([\mathbf{S}])$.

E x a m p l e 1. For decompositions by a system of orthogonal functions (Fourier transform, wavelet transform, etc.) the condition of admissible grouping is that resultant primitives have to be orthogonal. Each column of **[S]** can contain only one unit.

E x a m p l e 2. Pytiev morphological shape [3] is the projective orthogonal decomposition

$f(x, y) = \sum_{i = 1,\ldots,n} (a_i \bullet \chi_i(x, y))$,

where $\chi_i \in \{0,1\}$ is the characteristic function of the i-th image region with intensity a_i.

Suppose that the initial complete decomposition was a *trivial pixel tessellation*

$f(x, y) = \Sigma_{ij} (a_{ij} \bullet \varphi(i, j, x, y))$,

where $\varphi(i, j, x, y)$ is the *characteristic function of pixel*, $<x, y>$ - pixel position; $a_{ij} = f(i, j)$ – value of digital image at point $<i, j>$. Each column of the transition matrix should contain only one unit. But there is another *semantic constraint*: regions of image tessellation have to be *connected* due to the fact that the aggregation of regions without the common boundary provides no gain in descriptor size. So, the segmentation scheme without loss based on pixel tessellation automatically generates Pytiev morphological shapes.

E x a m p l e 3. Bynary Serra's Mathematical morphology (MM) [4] can be described as a *monotonous morphological decomposition* ('V'='∪'). Let the initial complete decomposition be a set of parameterized structuring elements $\mathbf{X_B} = \{B(x, y, R)\}$, where $B(0, 0, 1)$ is the *reference pattern*; $\langle x, y\rangle$ is a *position* and $R \in [0,R_{\max})$ is a *scale* of structuring element $B(x, y, R)$. $B(x, y ,0)$ is a single pixel with coordinates $\langle x, y\rangle$.

The morphological projector (*opening*) based on complete decomposition with $\mathbf{X_B}$ basis is described as

$$Pr(A,\mathbf{X_B}) = \cup_{B(x,y,R)\in \mathbf{X_B}} \{Pr(A,B(x,y,R))\} = \\ =\cup_{B(x,y,R)\in \mathbf{X_B}} \{B(x,y,R): B(x,y,R)\subseteq A\} \qquad (9)$$

Let us consider the problem of searching for optimal subbasis $\mathbf{Y_B}$ in complete decomposition $\mathbf{X_B}$. In case of binary MM, there is another natural method for effective joining of primitives: *greater elements can absorb smaller ones that completely belong to them*. Thus, the segmentation problem can be reduced to the problem of finding of *minimal number of primitives* $B(x, y, R)$ to be united into the pattern A. In particular, the solution of this problem for binary images and disk structuring elements gives the well-known procedure of *morphological skeletonization* [5]. So, we can conclude that described segmentation scheme also generates a wide class of *skelitonization* techniques.

Projective morphologies based on segmentation with loss of information

Let us consider the morphological segmentation operator (6) based on subbasis $\mathbf{Y}$ such that $Pr(A,\mathbf{Y})\neq A$. Such segmentation procedure can be called segmentation *with loss of information*.

E x a m p l e 4. Simplest *projective morphological compression* with loss includes the quantization of coefficients of decomposition (1) and following packing of them using any standard lossless compression methods. Since *quantization* is a projective operator, thus such "compression" is an algebraic projector and morphological filter.

Let us define a wider class of such projectors.

At first, consider *minimal distance projectors*

$$\Phi(A,\mathbf{Y}) = \|A - Pr(A,\mathbf{Y})\| + \alpha \times \nu(\mathbf{d}(A,\mathbf{Y})),$$

$$\Phi(A,\mathbf{Y}) \rightarrow min(\mathbf{Y}: \boldsymbol{Y}\subseteq \boldsymbol{X}), \qquad (10)$$

where *distance* $\rho(A,B) = \|A–B\|$ satisfies the properties

$\forall A, B, C \in \Omega: \rho(A,B) \geq 0, \rho(A,A) = 0,$

$\rho(A,B) + \rho(B,C) \leq \rho(A,C)$.

Additionally assume that for any basis $\mathbf{E}$

$$\forall A \in \Omega: B = Pr(A,\mathbf{E}) \Leftrightarrow B \in \boldsymbol{E}, \\ \forall C \in \boldsymbol{E}: \|A - B\| \leq \|A - C\|. \qquad (11)$$

Projectors satisfying (10)-(11) called *minimum-distance projectors*. In particular, projectors in Examples 1, 2 are of this class.

P r o p o s i t i o n 1. For minimum-distance projectors (11), the procedure of morphological segmentation (7), (10) defines a projective morphological filter based on segmentation (5).

This proposition was proved in [6]. So, we can speak about a class of morphological systems

<**X**,p(**X**→**Y**),α,ν,ρ> based on *distance* ρ, *basis* **X**, *transition rules* p(**X**→**Y**), *weight parameter* α and *descriptor size* ν(**d**(**Y**)).

E x a m p l e 5. Segmentation of 1D function using dynamic programming technique (*DP-segmentation*). 1D function $f(x)$ is given. 1D solution function $L(x) = Pr(f(x), \mathbf{Y}(x))$ is the piecewise-constant. Segmentation criterion is $\|L(x) - f(x)\| + \alpha N$, where N is a number of function segments. Such operator can be applied to 2D grayscale images via 2D→1D mapping, for example, by Peano scanning [7].

At second, consider *monotonous projectors*

$$\forall A, \mathbf{E}: Pr(A,\mathbf{E}) \le A \qquad (12)$$

$$\forall A, \mathbf{E}: Pr(A,\mathbf{E}) \ge A, \qquad (13)$$

where $A \le B \Leftrightarrow \forall x, y: A(x, y) \le B(x, y)$.
Projectors (12) and (13) called *opening* and *closing* operators respectively.
In general, the *monotonous projector* is any projector such that

$$\forall A, \mathbf{E}: V(Pr(A, \mathbf{E})) \subseteq V(A),$$

$$Pr(A, \mathbf{E}) \in V(Pr(A, \mathbf{E})). \qquad (14)$$

Here $V(A)$ is a *domain of admissible solution*s. Let *segmentation cost function* has a form

$$\Phi(A, \mathbf{Y}) = J(B) + \alpha \times \nu(\mathbf{d}(\mathbf{Y}(\mathrm{B}))),$$

$$\Phi(A, \mathbf{Y}) \to min(\mathbf{Y}: \boldsymbol{Y} \subseteq \boldsymbol{X},$$

$$B=Pr(A,\mathbf{Y}) \in V(A)), \qquad (15)$$

where $B = Pr(A,\mathbf{Y})$ is the *segmented pattern*; $\mathbf{Y}(B)$ – complete basis of B; $J(B)$ is the *reconstruction cost function*, depended only on the segmented pattern.
The following proposition is proved [6].
P r o p o s i t i o n 2. For monotonous projectors (14), in accordance with criterion (15), the morphological segmentation procedure (7) determines a morphological filter (5).
So, we can speak about morphological systems <**X**,p(**X**→**Y**),α,ν,*J*> based on *initial basis* **X**, *transition rules* p(**X** → **Y**), *weight parameter* α, *functional of descriptor size* ν(**d**(**Y**)), and *reconstruction cost function* $J(Pr(A,\mathbf{Y}))$.

E x a m p l e 6. Monotonous segmentation of 1D function using dynamic programming technique. The segmentation model is similar to one in Example 5 with account of (12) for *DP-opening* and (13) for *DP-closing*.
E x a m p l e 7. Morphological segmentation of binary dot patterns based on Hough transform. Following simple algorithm proposed [1]:
Step 1: Perform the Hough transform (HT);
Step 2: Binarize the Hough accumulator with fixed threshold value *t*;
Step 3: Delete all dots of source dot pattern those not belong to detected straight lines.
Such procedure defines a *Hough-projector* (MM opening filter). The descriptor size is directly determined by the number of nonzero elements of binarized accumulator. Thus, the segmentation problem is reduced to search of *optimal segmentation parameter* $t_{opt} = t(\alpha)$.

Conclusion

Problems of image segmentation were considered in the framework of projective morphology. Optimal segmentation operators are proved to be the algebraic projectors and determine projective morphologies based on segmentation. This scheme automatically generates Pytiev's morphological shapes and Serra's skeletonization technique. Some examples of segmentation projectors are considered, including segmentation based on dynamic programming and Hough transform.

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